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G-Function of Two Variables and the Conduction of Heat on a Flat Earth

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Abstract

Various problems in science and technology, when formulated mathematically, lead naturally to certain classes of partial differential equations involving one or more unknown functions together with the prescribed conditions (known as boundary conditions) which arise from the physical situation. Several workers have obtained solutions to the equations related to certain problems, which satisfy the given boundary conditions. The classical method in obtaining solutions of the boundary value problems of mathematical physics can be derived from Fourier's. Another technique using integral transforms, which had its origin in Heaviside's work, has been developed in the past and has certain advantages over the classical method.

The object of this paper is to evaluate an integral involving G-function of two variables and employ it to derive a solution of the problem of the conduction of heat in a flat earth.

Keywords: Boundary Value Problems, G-Function of Two Variables, Conduction of Heat, Flat Earth.

Introduction

The G-function of two variables was defined by Shrivastava and Joshi [4, p. 471] in terms of Mellin-Barnes type integrals as follows:

$$G^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[\begin{matrix}x\\y\end{matrix}\middle|\begin{matrix}(a_j;1,1)_{1,p_1}:(c_j,1)_{1,p_2}:(e_j,1)_{1,p_3}\\(b_j;1,1)_{1,q_1}:(d_j,1)_{1,q_2}:(f_j,1)_{1,q_3}\end{matrix}\right]$$

$$= \frac{-1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\theta_2(\xi)\theta_3(\eta)x^{\xi}y^{\eta}d\xi d\eta \qquad (1)$$

where

$$\phi_1(\xi,\eta) = \frac{\prod_{j=1}^{n_1}\Gamma(1-a_j+\xi+\eta)}{\prod_{j=n_1+1}^{p_1}\Gamma(a_j-\xi-\eta)\prod_{j=1}^{q_1}\Gamma(1-b_j+\xi+\eta)},$$

$$\theta_2(\xi) = \frac{\prod_{j=1}^{m_2}\Gamma(d_j-\xi)\prod_{j=1}^{n_2}\Gamma(1-c_j+\xi)}{\prod_{j=m_2+1}^{q_2}\Gamma(1-d_j+\xi)\prod_{j=n_2+1}^{p_2}\Gamma(c_j-\xi)}$$

$$\theta_3(\eta) = \frac{\prod_{j=1}^{m_3}\Gamma(f_j-\eta)\prod_{j=1}^{n_3}\Gamma(1-e_j+\eta)}{\prod_{j=m_3+1}^{q_3}\Gamma(1-f_j+\eta)\prod_{j=n_3+1}^{p_3}\Gamma(e_j-\eta)}$$

x and y are not equal to zero, and an empty product is interpreted as unity p_i, q_i, n_i and m_j are non negative integers such that $p_i \geq n_i \geq 0$, $q_i \geq 0$, $q_j \geq m_j \geq 0$, (i = 1, 2, 3; j = 2, 3).

The contour L_1 is in the ξ-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(d_j - \xi)$ (j = 1, ..., m_2) lie to the right, and the poles of $\Gamma(1 - c_j + \xi)$ (j = 1, ..., n_2), $\Gamma(1 - a_j + \xi + \eta)$ (j = 1, ..., n_1) to the left of the contour.

The countor L_2 is in the η-plane and runs from $-i\infty$ to $+i\infty$, with loops, if necessary, to ensure that the poles of $\Gamma(f_j - \eta)$ (j = 1, ..., m_3) lie to the right, and the poles of $\Gamma(1 - e_j + \eta)$ (j = 1, ..., n_3), $\Gamma(1 - a_j + \xi + \eta)$ (j = 1, ..., n_1) to the left of the contour.

and the double integral converges if

$$2(n_1 + m_2 + n_2) > (p_1 + q_1 + p_2 + q_2)$$
$$2(n_1 + m_3 + n_3) > (p_1 + q_1 + p_3 + q_3)$$

and $\quad |\arg x| < \tfrac{1}{2}U\pi, \ |\arg y| < \tfrac{1}{2}V\pi$

where $\quad U = [n_1 + m_2 + n_2 - \tfrac{1}{2}(p_1 + q_1 + p_2 + q_2)]$

$$V = [n_1 + m_3 + n_3 - \tfrac{1}{2}(p_1 + q_1 + p_3 + q_3)]$$

These assumptions for the G-function of two variables will be adhered to throughout this research work.

The following formulae are required in the proof:

Legendre's duplication formula [3]:

$\sqrt{\pi}\Gamma(2z) = 2^{2Z-1}\Gamma(z)\,\Gamma(z + ½)$. (2)

The following modified form of the integral [2]:

$$\int_{-T}^{T} e^{-2\pi imt/T}\left(\cos\frac{\pi t}{2T}\right)^{\omega-1} dt = \frac{T\Gamma(\omega)}{2^{\omega-2}\Gamma\left(\frac{\omega-4m+1}{2}\right)\Gamma\left(\frac{\omega+4m+1}{2}\right)}, \mathrm{Re}(\omega) > -1. \quad (3)$$

Aim of the Study

To derive a solution of the problem of the conduction of heat in a flat earth involving G-function of two variables.

Integral

The integral, which we need as follows:

$$\int_{-T}^{T} e^{-2\pi imt/T}\left(\cos\frac{\pi t}{2T}\right)^{\omega-1} G^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[{}^{z_1\left(\cos\frac{\pi t}{2T}\right)^2}_{z_2}\Big|{}^{(a_j;1,1)_{1,p_1}:(c_j,1)_{1,p_2}:(e_j,1)_{1,p_3}}_{(b_j;1,1)_{1,q_1}:(d_j,1)_{1,q_2}:(f_j,1)_{1,q_3}}\right]dt$$

$$= \frac{2T}{\sqrt{\pi}} G^{0,n_1:m_2,n_2+2;m_3,n_3}_{p_1,q_1:p_2+2,q_2+2;p_3,q_3}\left[{}^{z_1}_{z_2}\Big|{}^{(a_j;1,1)_{1,p_1}:\left(1-\frac{\omega}{2},1\right),\left(\frac{1}{2}-\frac{\omega}{2},1\right),(c_j,1)_{1,p_2}:(e_j,1)_{1,p_3}}_{(b_j;1,1)_{1,q_1}:(d_j,1)_{1,q_2},\left(\frac{1}{2}-\frac{\omega\pm4m}{2},1\right):(f_j,1)_{1,q_3}}\right], \quad (4)$$

provided that

$\omega > 0$, $|\arg x| < ½\,U\pi$, $|\arg y| < ½\,V\pi$

where $U = [n_1 + m_2 + n_2 - ½\,(p_1 + q_1 + p_2 + q_2)]$

$V = [n_1 + m_3 + n_3 - ½\,(p_1 + q_1 + p_3 + q_3)]$

Proof

The integral (4) can be obtained easily by making use of the definition of G-function of two variables as given in (1) and the formulae (2) and (3).

Formulate the Problem

The surface of the earth is considered as a plane and f(t) is assumed as an averaged purely periodic temperature (annual averaged or daily averaged temperature). In order to determine the temperature in the interior of the earth ignoring the increase of temperature in the interior of the earth due to radio active or nuclear processes. We can, in general, use the method given in Sommerfeld [6] by setting $x_0 = 0$ (surface of the earth), $x_1 = \infty$ (great depth) and $u_0 = f(t)$ for x = 0. It is convenient in this case to expand f(t) into a complex Fourier series:

$f(t) = \sum_{n=-\infty}^{\infty} C_n\, e^{2\pi int/T}$, (T = length of year or day) (5)

and to set for temperature in the interior of the earth at a depth x:

$u(x, t) = \sum_{n=-\infty}^{\infty} C_n\, u_n(x) e^{2\pi int/T}$ (6)

Each individual term of the series must satisfy the basic equation of heat conduction, viz., $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$. Thus yields the ordinary differential equation for u_n.

$\frac{d^2u_n}{dx^2} = p_n^2 u_n$ where $p_n^2 = \frac{2\pi in}{kT}$ (7)

In order for (6) to go into (5) for x = 0, we must have $u_n(0) = 1$.

Depending on whether n is positive or negative, we take

$2in = (1 + i)^2|n|$ (8)

and

$p_n = (1 + i)q_n,\ q_n = \sqrt{\frac{|n|\pi}{kT}} > 0.$ (9)

The general solution of (7) is obtained as:

$u_n(x) = A_n e^{(1+i)q_n x} + B_n e^{-(1+i)q_n x}$. (10)

In equation (10), we must have $A_n = 0$, since otherwise the temperature would become infinite for $n \to \infty$ and $B_n = 1$, to satisfy (8). Substituting this in (6), we obtain our solution as [4]:

$u(x, t) = \sum_{n=-\infty}^{\infty} C_n\, e^{-(1+i)q_n x} e^{2\pi int/T}$. (11)

In this paper, we have

$$u(0, t) = f(t) = \left(\cos\frac{\pi t}{2T}\right)^{\omega-1} G^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[{}^{z_1\left(\cos\frac{\pi t}{2T}\right)^2}_{z_2}\Big|{}^{(a_j;1,1)_{1,p_1}:(c_j,1)_{1,p_2}:(e_j,1)_{1,p_3}}_{(b_j;1,1)_{1,q_1}:(d_j,1)_{1,q_2}:(f_j,1)_{1,q_3}}\right] \quad (12)$$

Methodology

Beg [1], Shrivastava [5] and several other authors have obtained solutions of various boundary value problems involving G-Function.

Following Beg [1], Shrivastava [5] and several other authors, we will employ the G-function of two variables in obtaining a solution of a boundary value problem related to conduction of heat in a flat earth, which will be useful for further research.

Solution of the Problem

The solution of the problem to be obtained is

$$u(x, t) = \frac{1}{\sqrt{\pi}}\sum_{n=-\infty}^{\infty} e^{-(1+i)q_n x}\, e^{2\pi int/T} \times$$

$$G^{0,n_1:m_2,n_2+2;m_3,n_3}_{p_1,q_1:p_2+2,q_2+2;p_3,q_3}\left[{}^{z_1}_{z_2}\Big|{}^{(a_j;1,1)_{1,p_1}:\left(1-\frac{\omega}{2},1\right),\left(\frac{1}{2}-\frac{\omega}{2},1\right),(c_j,1)_{1,p_2}:(e_j,1)_{1,p_3}}_{(b_j;1,1)_{1,q_1}:(d_j,1)_{1,q_2},\left(\frac{1}{2}-\frac{\omega\pm4n}{2},1\right):(f_j,1)_{1,q_3}}\right] \quad (13)$$

where all conditions of convergence are same as in (4).

Proof

If t = 0, then by virtue of (11) and (12), we have

$$\sum_{n=-\infty}^{\infty} C_n\, e^{2\pi int/T} = \left(\cos\frac{\pi t}{2T}\right)^{\omega-1} G^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[{}^{z_1\left(\cos\frac{\pi t}{2T}\right)^2}_{z_2}\Big|{}^{(a_j;1,1)_{1,p_1}:(c_j,1)_{1,p_2}:(e_j,1)_{1,p_3}}_{(b_j;1,1)_{1,q_1}:(d_j,1)_{1,q_2}:(f_j,1)_{1,q_3}}\right] \quad (14)$$

Multiplying both side of (14) by $e^{-2\pi imt/T}$ and integrating with respect to t from − T to T, we get

$$\int_{-T}^{T} e^{-2\pi imt/T}\left(\cos\frac{\pi t}{2T}\right)^{\omega-1} \times G^{0,n_1:m_2,n_2;m_3,n_3}_{p_1,q_1:p_2,q_2;p_3,q_3}\left[{}^{z_1\left(\cos\frac{\pi t}{2T}\right)^2}_{z_2}\Big|{}^{(a_j;1,1)_{1,p_1}:(c_j,1)_{1,p_2}:(e_j,1)_{1,p_3}}_{(b_j;1,1)_{1,q_1}:(d_j,1)_{1,q_2}:(f_j,1)_{1,q_3}}\right]dt$$

$$= \sum_{n=-\infty}^{\infty} C_n \int_{-T}^{T} e^{2\pi i(n-m)t/T}\, dt.$$

Now using (4) and orthogonality property of the exponential functions, viz.

$$\int_{-T}^{T} e^{2\pi i(n-m)t/T}\,dt = \begin{cases} 0, & \text{if } m \neq n; \\ 2T, & \text{if } m = n; \end{cases}$$

we get

$$C_m = \frac{1}{\sqrt{\pi}} G^{0,n_1:m_2,n_2+2;m_3,n_3}_{p_1,q_1:p_2+2,q_2+2;p_3,q_3}\left[\begin{matrix} z_1 \\ z_2 \end{matrix}\middle|\begin{matrix} (a_j;1,1)_{1,p_1}:\left(1-\frac{\omega}{2},1\right),\left(\frac{1}{2}-\frac{\omega}{2},1\right),(c_j,1)_{1,p_2}:(e_j,1)_{1,p_3} \\ (b_j;1,1)_{1,q_1}:(d_j,1)_{1,q_2},\left(\frac{1}{2}-\frac{\omega\pm 4m}{2},1\right):(f_j,1)_{1,q_3} \end{matrix}\right] \tag{15}$$

With the help of (11) and (15), the solution (13) is obtained.

Conclusion

Since the G–functions play a crucial role in a certain useful mathematical enterprise. When looked at conceptually, they are both natural and attractive. Most special functions, and many products of special functions, are G–functions or are expressible as products of G–functions with elementary functions.

On specializing the parameters, G-function of two variables may be reduced to several other higher transcendental functions. Therefore the result (13) is of general nature and may reduce to be in different forms, which will be useful in the literature on applied Mathematics and other branches.

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